MATH 312
Lecture 25

LINEAR PROGRAMMING and DUALITY

So far, we have focused on linear equations of the form $Ax = b$ or maybe $Ax = \lambda x$. But for various practical reasons, one often has to work with linear INEQUALITIES of the form $Ax \leq b$; particularly in economics.

LINEAR INEQUALITIES

While linear equations (eg in 3 variables) look like

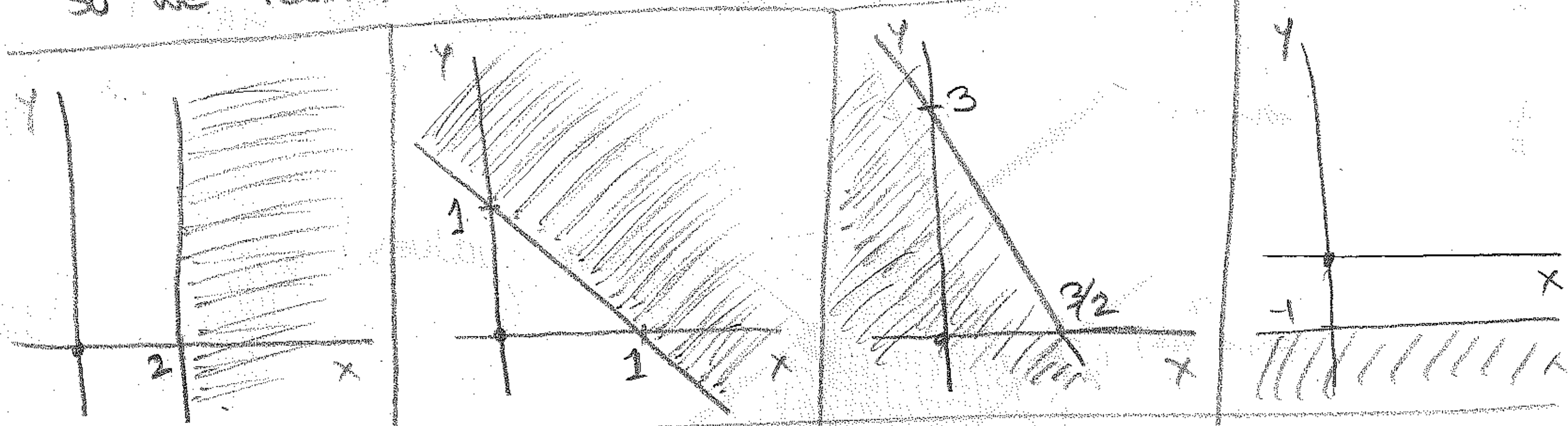
$$ax + by + cz = d, \qquad (a, b, c, d \text{ scalars})$$

linear inequalities look like:

$$\boxed{ax + by + cz \leq d}$$

This is the same as $-ax - by - cz \geq -d$, so no need to include $\geq$ cases seperately...

Each linear inequality carves out a "half-space". For visualizing, it is easiest to look at $\mathbb{R}^2$, so we restrict to 4 examples of inequalities in 2 variables.



$x \geq 2$	$x + y \geq 1$	$2x + y \leq 3$	$y \leq -1$

The basic idea is that

[INEQUALITIES MODEL CONSTRAINTS]

Particularly, constraints arising in standard economics-related situations like this:

Eg I You are given 1 dollar, and asked to invest some fraction of it in assets A and B, so that

- At least twice as much is invested in A as in B, and
- At least ½ a dollar is invested somewhere.

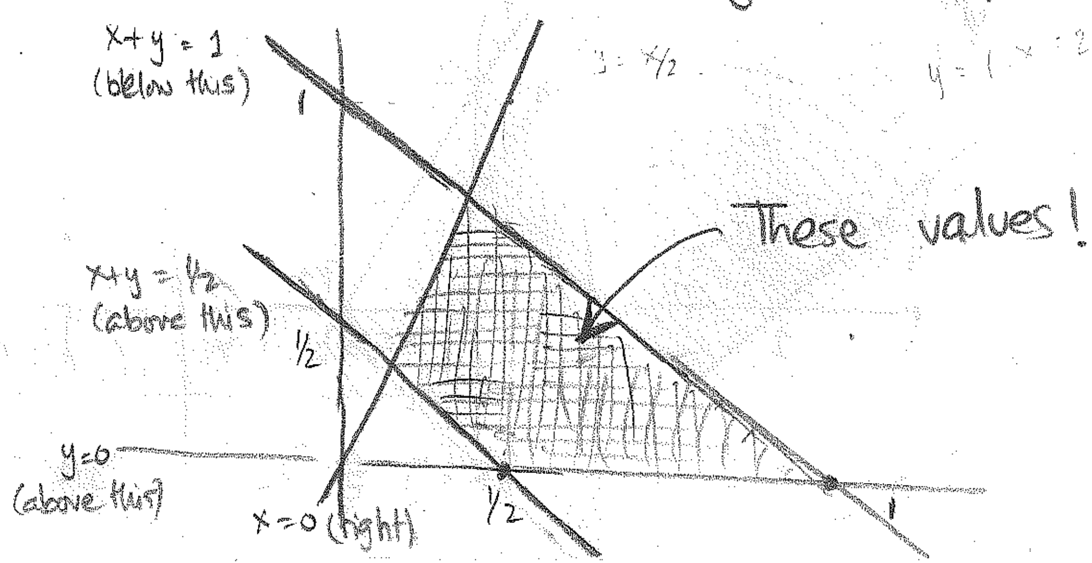
So, if you invest x in A and y in B, there are 5 linear inequalities floating around:

- $x + y \leq 1$ (total = 1)
- $x \geq 2y$ (twice as much in x...)
- $x + y \geq \frac{1}{2}$ (at least ½ invested)

and of course,
- $x \geq 0$ (can't invest negative \$)
- $y \geq 0$

Q: What values are x and y allowed to take?

A



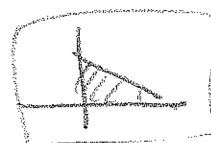
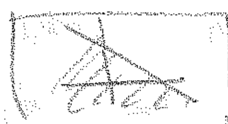
Note that the collection of 5 inequalities can be encoded with a matrix inequality:

$$A\vec{x} \le \vec{b}, \quad \text{i.e.}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}$$

<u>Def</u> The $\boxed{\text{FEASIBLE REGION}}$ of $Ax \le b$ is the "common intersection" of the halfspaces arising from the individual inequalities.

The Feasible Region may be

- Bounded $\longrightarrow$ ($x+y \le 1$, $x \geq 0$, $y \geq 0$) 
- Unbounded $\rightarrow$ ($x+y \le 1$) 
- Empty $\rightarrow$ ($x+y \geq 1$, $x+y \le 0$) ??

LINEAR PROGRAMMING

<u>Def</u> A LINEAR PROGRAMMING problem asks that we MAXIMIZE $\vec{c}^T\vec{x}$ subject to $A\vec{x} \le \vec{b}$ and $\vec{x} \geq 0$: here $\vec{c}$, A and $\vec{b}$ are given but $\vec{x}$ is to be determined...

So, back to our $\boxed{\text{Eg I}}$ with 1 dollar of investment: let's say investment in A has a twofold return while investment in B is threefold.

If we invest x in A and y in B, we get back

$$3x + 2y = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Set $\vec{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, so this is $\vec{c}^T \begin{bmatrix} x \\ y \end{bmatrix}$.

Next, consider only the three (out of 5 inequalities) which are NOT $x \geq 0$ and $y \geq 0$. These can be written as

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}$$

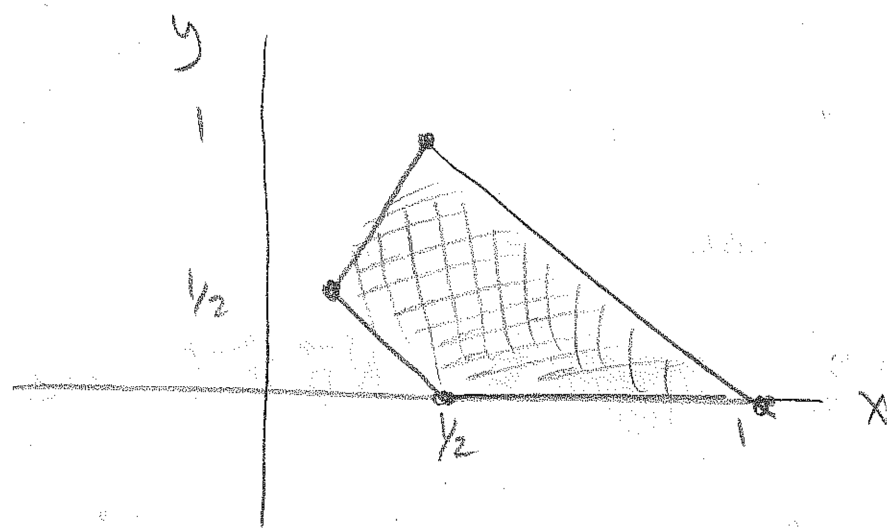
So, maximizing return-on-investment in $\boxed{\text{Eg I}}$ brings us to the following Linear Program:

$$\begin{bmatrix} \text{MAXIMIZE} \\ \vec{c}^T \vec{x} \end{bmatrix} \quad \begin{bmatrix} \text{SUBJECT TO} \\ A\vec{x} \leq \vec{b} \end{bmatrix} \text{ and } \vec{x} \geq 0$$

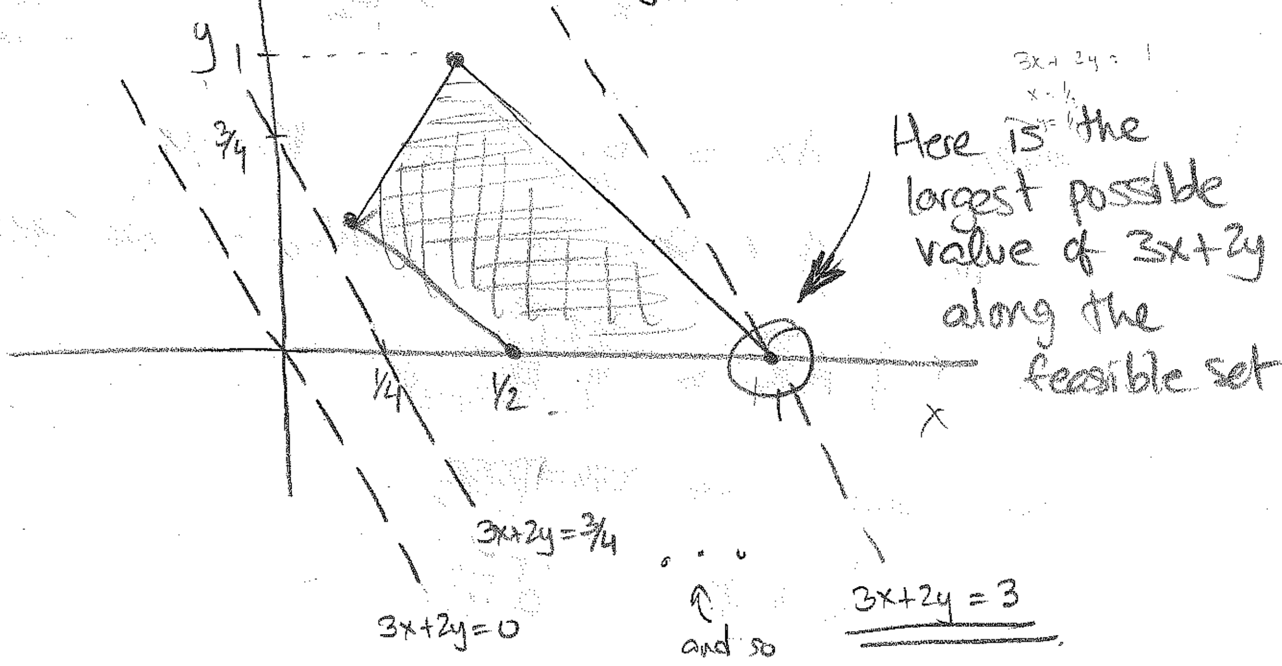
where $\vec{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}$. We have to <u>find</u> the

OPTIMAL $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}$

This particular problem admits a nice, geometric solution: draw the feasible set:



And the lines $3x+2y =$ constant: for many constants: just slide it along the feasible set.



NOTE The optimum will "always" occur at a a CORNER of the feasible set: so, if one has this set, it is enough to evaluate $c^T \vec{x}$ on its corners!

DUALITY

The LP, (call it the PRIMAL problem)

$\underline{\text{Max}}\ \vec{c}^T\vec{x}$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$

has a DUAL LP,

$\underline{\text{Min}}\ \vec{b}^T\vec{p}$ subject to $A^T\vec{p} \geq \vec{c}$ and $\vec{p} \geq 0$

<u>Thm</u> [Weak duality] Given any solutions $\vec{x}_*$ to PRIMAL and $\vec{p}_*$ to DUAL, we have

$$\vec{c}^T\vec{x}_* \leq \vec{b}^T\vec{p}_*.$$

[Max primal solution is smaller than min dual solution: If both exist!]

<u>Pf</u> Given $A\vec{x} \leq \vec{b}$ in PRIMAL, and $\vec{y} \geq 0$ in DUAL, we get

$$\vec{y}^TA\vec{x} \leq \vec{y}^T\vec{b}$$

But now, we transpose

$$\vec{x}^TA^T\vec{y} \leq \vec{b}^T\vec{y}$$

By DUAL, $A^T\vec{y} \geq \vec{c}$, so

$$\underset{\underline{\vec{c}^T\vec{x}}}{\overset{\|}{\vec{x}^T\vec{c}}} \leq \underline{\vec{b}^T\vec{y}}$$

Done.